C4 Integration Answers

Specimen

8. (a) $\int_0^{\frac{\pi}{4}} \cos^2\theta \, d\theta = \int_0^{\frac{\pi}{4}} \frac{1+\cos 2\theta}{2} d\theta$ — M1 ($a + b\cos 2\theta$)

A1 ($a = ½$, $b = ½$)

$= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{4}}$ — B1

$= \frac{\pi}{8} + \frac{1}{4} - \frac{0}{2} - \frac{0}{4}$

$= \frac{\pi}{8} + \frac{1}{4}$ — A1 (convincing)

(b) $x = 3\tan\theta$, $dx = 3\sec^2\theta \, d\theta$ — M1 (substitution for dx)

$x = 0$, $\theta = 0$, $x = 3$, $\theta = \frac{\pi}{4}$ — B1 (limits)

$27\int_0^{\frac{\pi}{4}} \frac{3\sec^2\theta}{(9\tan^2\theta + 9)^2} d\theta = 27\int_0^{\frac{\pi}{4}} \frac{3\sec^2\theta}{81\sec^4\theta} d\theta$ — A1 (unsimplified)

A1 ($1 + \tan^2\theta = \sec^2\theta$)

$= \frac{3 \times 27}{81} \int_0^{\frac{\pi}{4}} \cos^2\theta d\theta$ — A1 (simplified)

$= \left(\frac{\pi}{8} + \frac{1}{4}\right)$ — A1

2005 Summer

7. (a) $u = 2x - 1$, $du = 2\,dx$

when $x = 0$, $u = -1$; $x = 1$, $u = 1$

$\int_{-1}^{1} \frac{(u+1)}{2} u^9 \frac{du}{2}$ — M1 A1 (no limits required)

$= \frac{1}{4}\int_{-1}^{1} (u^{10} + u^9) du$ — m1

$= \frac{1}{4}\left[\frac{u^{11}}{11} + \frac{u^{10}}{10}\right]_{-1}^{1}$ — A1

$= \frac{1}{22}$ — A1 (F.T. one slip)

(b)	(i)	$\int x\cos 2x\,dx = \dfrac{x\sin 2x}{2} - \int \dfrac{\sin 2x}{2}\,dx$	M1 A1 A1
		$= \dfrac{x\sin 2x}{2} + \dfrac{\cos 2x}{4}\ (+C)$	A1 (F.T. one slip)
	(ii)	$\int x\cos^2 x\,dx = \int x\dfrac{(1+\cos 2x)}{2}\,dx$	M1 A1
		$= \dfrac{x^2}{4} + \dfrac{x\sin 2x}{4} + \dfrac{\cos 2x}{8}\ (+C)$	A1 (F.T. first result) [12]

2006 Summer

7.	*(a)*		$\int x\ln x\,dx = \dfrac{x^2}{2}\ln x - \int \dfrac{x^2}{2}.\dfrac{1}{x}\,dx$	M1 $(f(x)\ln x - \int g(x).\frac{1}{x}dx)$
				A1($f(x)=g(x)$) A1 ($f(x) = g(x)=\frac{x^2}{2}$)
		$=$	$\dfrac{x^2}{2}\ln x - \int \dfrac{x}{2}\,dx$	A1
		$=$	$\dfrac{x^2}{2}\ln x - \dfrac{x^2}{4}(+C)$	A1 (C.A.O.)
	(b)	$u =$	$2\sin x + 3,\ \ dx = 2\cos x dx$ (limits are 3, 4)	
			$\int_3^4 \dfrac{1}{2}\dfrac{du}{u^2}$	M1 $\left(\int \frac{a}{u^2}du \text{ with } a = \pm\frac{1}{2}, 1, 2\right)$
				A1 ($a = \frac{1}{2}$)
		$=$	$\left[-\dfrac{1}{2u}\right]_3^4$	A1 ($-\frac{a}{u}$, allowable a)
		$=$	$\dfrac{1}{24}$ (o.e.)	A1 (F.T. allowable a or one slip)

2007 Summer

7.	*(a)*	$\int x^2\ln x\,dx = \dfrac{x^3\ln x}{3} - \int \dfrac{x^3}{3}.\dfrac{1}{x}\,dx$	M1 (Parts and correct choice of u, v)
		$= \dfrac{x^3\ln x}{3} - \int \dfrac{x^2}{3}\,dx$	A1
			M1 (division)
		$= \dfrac{x^3\ln x}{3} - \dfrac{x^3}{9}\quad (+C)$	A1 (C.A.O.)

(b) $dx = 2\cos\theta d\theta$

When $x = 0,\ \theta = 0$

$x = \sqrt{2},\ \sin\theta = \frac{1}{\sqrt{2}},\ \theta = \frac{\pi}{4}$ — B1

$$\int_0^{\frac{\pi}{4}} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}.2\cos\theta d\theta$$

M1 (substitution for dx and x)

A1 (any limits)

$$= \int_0^{\frac{\pi}{4}} \frac{4\sin^2\theta}{2\sqrt{1-\sin^2\theta}}.2\cos\theta d\theta$$

2008 Summer

6. *(a)* $\int (3x+1)e^{2x}dx = (3x+1)\frac{e^{2x}}{2} - \int \frac{3}{2}e^{2x}dx$

M1 ($f(x)(3x+1) - \int 3f(x)dx$)

A1 ($f(x) = ke^{2x}, k = 1, \frac{1}{2}, 2$)

A1 ($k = \frac{1}{2}$)

$$= (3x+1)\frac{e^{2x}}{2} - \frac{3e^{2x}}{4}(+C)$$

A1 (F.T. one slip)

(b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}.3\cos\theta d\theta$

B1 (for first line, unsimplified)

B1 (simplified without limits)

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 9\cos^2\theta d\theta$$

B1 (limits)

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)d\theta$$

M1 ($\cos^2\theta = a + b\cos 2\theta$)

A1 ($a = b = \frac{1}{2}$)

$$= \left[\frac{9\theta}{2} + \frac{9\sin 2\theta}{4}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

M1 ($k\sin 2\theta, k = \pm\frac{b}{2}, 2b, b$)

A1 ($k = \frac{b}{2}$)

$$= \frac{3\pi}{2} - \frac{9\sqrt{3}}{8} \quad or \quad 2.764$$

A1 (C.A.O.)

2009 Summer

6. *(a)* $\int (x+3)e^{2x}\,dx = (x+3)\dfrac{e^{2x}}{2} - \int 1.e^{2x}\,dx$

$((x+3)f(x) - \int Af(x)dx;\ f(x) \neq k,\ A = 1, 3)$ M1

$(f(x) = ke^{2x})$ A1

$\left(k = \dfrac{1}{2},\ A = 1\right)$ A1

$= (x+3)\dfrac{e^{2x}}{2} - \dfrac{e^{2x}}{4} + C$ C.A.O. (must contain C) A1

(b) $\int_3^2 -\dfrac{1}{2u^{\frac{3}{2}}}\,du$ $\left(\dfrac{k}{u^{\frac{3}{2}}}\right)$ M1

$\left(k = -\dfrac{1}{2}\right)$ A1

$= \left[-u^{\frac{1}{2}}\right]_3^2$ (integration, any k, no limits) A1

(correct use of limits) m1

$= \sqrt{2} + \sqrt{3} \approx 0\cdot318$ C.A.O. (either answer) A1

Answer only gains 0 marks

2010 Summer

7. (a) $\int x^3 \ln x\,dx = f(x)\ln x - \int f(x)\,g(x)\,dx$ M1

$f(x) = \dfrac{x^4}{4},\ g(x) = \dfrac{1}{x}$ A1, A1

$\int x^3 \ln x\,dx = \dfrac{x^4}{4}\ln x - \dfrac{x^4}{16} + c$ (c.a.o.) A1 (*b*) $\int x(2x-3)^4\,dx = \int f(u) \times u^4 \times k\,du$

$(f(u) = pu + q,\ p \neq 0,\ q \neq 0$ and $k = {}^1/_2$ or $2)$ M1

$\int x(2x-3)^4\,dx = \int \frac{(u+3)}{2} \times u^4 \times \frac{du}{2}$ A1

$\int (au^5 + bu^4)\,du = \frac{au^6}{6} + \frac{bu^5}{5}$ $(a \neq 0,\ b \neq 0)$ B1

Either: Correctly inserting limits of –1, 1 in candidate's $\frac{au^6}{6} + \frac{bu^5}{5}$

or: Correctly inserting limits of 1, 2 in candidate's $\frac{a(2x-3)^6}{6} + \frac{b(2x-3)^5}{5}$ m1

$\int_1^2 x(2x-3)^4\,dx = \frac{3}{10}$ (c.a.o.) A1

2011 Summer

7. (*a*) $\int x \sin 2x\,dx = x \times k \times \cos 2x - \int k \times \cos 2x \times g(x)\,dx$

$(k = \pm{}^1/_2, \pm 2$ or $\pm 1)$ M1

$k = -\frac{1}{2},\ g(x) = 1$ A1, A1

$\int x \sin 2x\,dx = -\frac{1}{2} \times x \times \cos 2x + \frac{1}{4} \times \sin 2x + c$ (c.a.o.) A1

(*b*) $\int \frac{x}{(5-x^2)^3}\,dx = \int \frac{k}{u^3}\,du$ $(k = \pm{}^1/_2$ or $\pm 2)$ M1

$\int \frac{a}{u^3}\,du = -\frac{a}{2}u^{-2}$ B1

$\int_0^2 \frac{x}{(5-x^2)^3}\,dx = -\frac{k}{2}\left[u^{-2}\right]_5^1$ or $-\frac{k}{2}\left[\frac{1}{(5-x^2)^2}\right]_0^2$

(f.t. candidate's value for k, $k = \pm{}^1/_2$ or ± 2) A1

$\int_0^2 \frac{x}{(5-x^2)^3}\,dx = \frac{6}{25}$ (c.a.o.) A1

Note: Answer only with no working earns 0 marks

2012 Summer

7. (*a*)

$u = x \Rightarrow du = dx$ (o.e.) B1

$dv = e^{-2x}\,dx \Rightarrow v = -\frac{1}{2}e^{-2x}$ (o.e.) B1

$\int x e^{-2x}\,dx = x \times -\frac{1}{2}e^{-2x} - \int -\frac{1}{2}e^{-2x}\,dx$ M1

$\int x e^{-2x}\,dx = -\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x} + c$ (c.a.o.) A1

(*b*)

$\int \frac{1}{x(1+3\ln x)}\,dx = \int \frac{k}{u}\,du$ ($k = \frac{1}{3}$ or 3) M1

$\int \frac{a}{u}\,du = a \ln u$ B1

$\int_1^e \frac{1}{x(1+3\ln x)}\,dx = k[\ln u]_1^4$ or $k[\ln(1+3\ln x)]_1^e$ B1

$\int_1^e \frac{1}{x(1+3\ln x)}\,dx = 0{\cdot}4621$ (c.a.o.) A1

2013 Summer

7. (*a*)

$u = 3x - 1 \Rightarrow du = 3dx$ (o.e.) B1

$dv = \cos 2x\,dx \Rightarrow v = \frac{1}{2}\sin 2x$ (o.e.) B1

$\int (3x-1)\cos 2x\,dx = (3x-1) \times \frac{1}{2}\sin 2x - \int \frac{1}{2}\sin 2x \times 3dx$ M1

$\int (3x-1)\cos 2x\,dx = \frac{1}{2}(3x-1)\sin 2x + \frac{3}{4}\cos 2x + c$ (c.a.o.) A1 (*b*) $\displaystyle\int \frac{x}{(2x+1)^3}\,dx = \int \frac{f(u)}{u^3} \times k\,du$

($f(u) = pu + q$, $p \neq 0$, $q \neq 0$ and $k = {}^1/_2$ or 2) **M1**

$\displaystyle\int \frac{x}{(2x+1)^3}\,dx = \int \frac{(u-1)}{2} \times \frac{1}{u^3} \times \frac{du}{2}$ **A1**

$\displaystyle\int (au^{-2} + bu^{-3})\,du = \frac{au^{-1}}{-1} + \frac{bu^{-2}}{-2}$ ($a \neq 0$, $b \neq 0$) **B1**

Either: Correctly inserting limits of 1, 3 in candidate's $cu^{-1} + du^{-2}$
($c \neq 0$, $d \neq 0$)

or: Correctly inserting limits of 0, 1 in candidate's
$c(2x+1)^{-1} + d(2x+1)^{-2}$ ($c \neq 0$, $d \neq 0$) **m1**

$\displaystyle\int_0^1 \frac{x}{(2x+1)^3}\,dx = \frac{1}{18}$ ($= 0{\cdot}055\ldots$) (c.a.o.) **A1**

Note: **Answer only with no working earns 0 marks**

2014 Summer

7. (*a*) $u = \ln 2x \Rightarrow du = 2 \times \frac{1}{2x}\,dx$ (o.e.) **B1**

$dv = x^4\,dx \Rightarrow v = \frac{1}{5}x^5$ (o.e.) **B1**

$\displaystyle\int x^4 \ln 2x\,dx = \ln 2x \times \frac{1}{5}x^5 - \int \frac{1}{5}x^5 \times \frac{1}{x}\,dx$ (o.e.) **M1**

$\displaystyle\int x^4 \ln 2x\,dx = \ln 2x \times \frac{1}{5}x^5 - \frac{1}{25}x^5 + c$ (c.a.o.) **A1**

(*b*) $\displaystyle\int \sqrt{(10\cos x - 1)}\,\sin x\,dx = \int k \times u^{1/2}\,du$ ($k = -{}^1/_{10}$, ${}^1/_{10}$ or ± 10) **M1**

$\displaystyle\int a \times u^{1/2}\,du = a \times \frac{u^{3/2}}{3/2}$ **B1**

$\displaystyle\int_0^{\pi/3} \sqrt{(10\cos x - 1)}\,\sin x\,dx = k\left[\frac{u^{3/2}}{3/2}\right]_9^4$ or $\displaystyle k\left[\frac{(10\cos x - 1)^{3/2}}{3/2}\right]_0^{\pi/3}$ **B1**

$\displaystyle\int_0^{\pi/3} \sqrt{(10\cos x - 1)}\,\sin x\,dx = \frac{19}{15} = 1{\cdot}27$ (c.a.o.) **A1**

2015

7. (*a*) $\displaystyle\int \frac{x^2}{(12-x^3)^2}\,dx = \int \frac{k}{u^2}\,du$ $\quad (k = {}^1/_3, -{}^1/_3, 3 \text{ or } -3)$ M1

$\displaystyle\int \frac{a}{u^2}\,du = a \times \frac{u^{-1}}{-1}$ B1

Either: Correctly inserting limits of 12, 4 in candidate's bu^{-1}

or: Correctly inserting limits of 0, 2 in candidate's $b(12-x^3)^{-1}$ M1

$\displaystyle\int_0^2 \frac{x^2}{(12-x^3)^2}\,dx = \frac{1}{18}$ (c.a.o.) A1

Note: Answer only with no working earns 0 marks

(*b*) (i) $u = x \Rightarrow du = dx$ (o.e.) B1

$dv = \cos 2x\,dx \Rightarrow v = \frac{1}{2}\sin 2x$ (o.e.) B1

$\displaystyle\int x\cos 2x\,dx = x \times \frac{1}{2}\sin 2x - \int \frac{1}{2}\sin 2x \times dx$ M1

$\displaystyle\int x\cos 2x\,dx = \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$ (c.a.o.) A1

(ii) $\displaystyle\int x\sin^2 x\,dx = \int x\left(\frac{k}{2} - \frac{m}{2}\cos 2x\right)dx$ (o.e.)

$(k = 1, -1, m = 1, -1)$ M1

$\displaystyle\int x\sin^2 x\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$ A1

$\displaystyle\int x\sin^2 x\,dx = \frac{x^2}{4} - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c$

(f.t. only candidate's answer to (*b*)(*i*)) A1

2016

6. (*a*) $u = 2x + 1 \Rightarrow du = 2dx$ (o.e.) B1

$dv = e^{-3x}\,dx \Rightarrow v = -\frac{1}{3}e^{-3x}$ (o.e.) B1

$\displaystyle\int (2x+1)e^{-3x}\,dx = -\frac{1}{3}e^{-3x} \times (2x+1) - \int -\frac{1}{3}e^{-3x} \times 2dx$ (o.e.) M1

$\displaystyle\int (2x+1)e^{-3x}\,dx = -\frac{1}{3}e^{-3x} \times (2x+1) - \frac{2}{9}e^{-3x} + c$ (c.a.o.) A1 (*b*) $\displaystyle\int \frac{\sqrt{(4+5\tan x)}}{\cos^2 x}\,dx = \int k \times u^{1/2}\,du$ $\quad (k = {}^1/_5 \text{ or } 5)$ M1

$\displaystyle\int a \times u^{1/2}\,du = a \times \frac{u^{3/2}}{3/2}$ B1

Either: Correctly inserting limits of 4, 9 in candidate's $bu^{3/2}$

or: Correctly inserting limits of 0, $\pi/4$ in candidate's $b(4+5\tan x)^{3/2}$ M1

$\displaystyle\int_0^{\pi/4} \frac{\sqrt{(4+5\tan x)}}{\cos^2 x}\,dx = \frac{38}{15} = 2{\cdot}53$ (c.a.o.) A1

Note: Answer only with no working earns 0 marks

2017

7. (*a*) $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$ B1

$dv = x^{-4}\,dx \Rightarrow v = \frac{1}{-3}x^{-3}$ (o.e.) B1

$\displaystyle\int \frac{\ln x}{x^4}\,dx = \frac{1}{-3}x^{-3} \times \ln x - \int \frac{1}{-3}x^{-3} \times \frac{1}{x}\,dx$ (o.e.) M1

$\displaystyle\int \frac{\ln x}{x^4}\,dx = \frac{1}{-3}x^{-3} \times \ln x - \frac{1}{9}x^{-3} + c$ (c.a.o.) A1

(*b*) $\displaystyle\int x^3(x^2+1)^4\,dx = \int f(u) \times u^4 \times du$ $\quad (f(u) = pu + q,\ p \neq 0,\ q \neq 0)$ M1

$\displaystyle\int x^3(x^2+1)^4\,dx = \int \frac{(u-1)}{2} \times u^4 \times du$ A1

$\displaystyle\int (pu^5 + qu^4)\,du = \frac{pu^6}{6} + \frac{qu^5}{5}$ B1

Either: Correctly inserting limits of 1, 2 in candidate's $\frac{pu^6}{6} + \frac{qu^5}{5}$

or: Correctly inserting limits of 0, 1 in candidate's $\frac{p(x^2+1)^6}{6} + \frac{q(x^2+1)^5}{5}$ m1

$\displaystyle\int_0^1 x^3(x^2+1)^4\,dx = \frac{43}{20} = 2{\cdot}15$ (c.a.o.) A1